J. DIFFERENTIAL GEOMETRY
9 (1974) 475-482

ON CHERN'S KINEMATIC FORMULA IN INTEGRAL GEOMETRY

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Dedicated to S. S. Chern on his 60th birthday

1. Introduction

In 1939 Hermann Weyl [1] derived a formula for the volume of the tube of radius ρ about a compact manifold (without boundary) imbedded in a Euclidean space. The expression for this volume, for a manifold X of dimension k imbedded in a Euclidean n-space E^n is a polynomial $V(T_\rho^{(n)}(X))$ in ρ, valid for small ρ, when no self-intersections in the normal bundle occur. The coefficients of this polynomial are integrals over X of invariant polynomial functions of the Riemann-Chistoffel curvature tensor. The polynomial expression for the volume is of the form

$$V(T_\rho^{(n)}(X)) = \sum \gamma_{n,k,e}\mu_e(X)\rho^{n-k-e} , \tag{1.1}$$

where the summation extends over all even values of e such that $0 \leq e \leq k$. The $\mu_e(X)$ are the integral invariants referred to, while the $\gamma_{n,k,e}$ depend only on their subscripts and not on more subtle geometric properties of X. Thus γ and μ are uniquely determined up to a factor which depends on k and e. In what follows we add a superscript (1) to μ when quoting others.

In 1966 S. S. Chern [2] studied the same μ's from the point of view of the kinematic formula. Let M^p and M^q be compact manifolds of dimensions p and q imbedded in E^n, and let g be an element of the group of isometries in E^n. Then, for almost all $g, M^p \cap gM^q$ is again a manifold, and the $\mu_e^{(1)}(M^p \cap gM^q)$ are meaningful quantities. The kinematic formula of Chern deals with the integral $\int \mu_e^{(1)}(M^p \cap gM^q)d^{(1)}g$, where the integration extends over the group of isometries, and $d^{(1)}g$ is the Haar measure on this group, i.e., the product of the measure on E^n and that on the orthogonal group in n dimensions, the latter being a product of measures on spheres. This integral, according to Chern's theorem, is expressible as follows:

Communicated September 17, 1973. Research partially supported by an N.S.F. grant at the University of Pennsylvania.

$$\int \mu_e^{(1)}(M^p \cap gM^q)d^{(1)}g = \sum_{\substack{i+j=e \\ i,j \text{ even}}} c_{i,j,n,p,q}\mu_i^{(1)}(M^p)\mu_j^{(1)}(M^q) \ . \tag{1.2}$$

Again, the c's depend only on their subscripts and some normalization. Some of the hardest work in Chern's paper is devoted to a detailed calculation from which the c's can be determined.

The author's interest in Chern's work was stimulated by the fact that the right side of (1.2) depends bilinearly on the $\mu_i^{(1)}(M^p)$ and $\mu_j^{(1)}(M^q)$, and (at least to him) the suggestion of an underlying algebra with the c's as structure constants was inevitable. Simple formal properties of the integral in (1.2) support this initial impression: the interchange of p and q in (1.2) does not change the value (a simple change of variable in the integral shows this), so the "algebra" is commutative. Similarly, a formal manipulation of (1.2) shows that the "algebra" is associative.

A first step toward finding this "algebra" was a retracing of the c's; a second step a re-normalization of the μ's which would give the c's in (1.2) a simple form: we find the value 1 works.

The main result of this paper may be stated as follows:

Theorem I. *There exist a normalization of the μ's and a normalization of the Haar measure of the isometry group of E^n, as given by* (3.5), (3.6), (3.7), *such that the $c_{i,j,n,p,q}$ in* (1.2) *are equal to* 1.

Rephrased in terms of the "algebra" (which did not quite work out) the theorem says:

Theorem I′. *There exist a normalization of the μ's and a Haar measure dg on the isometry group of E^n, given by* (3.5), (3.6), (3.7), *so that the Chern curvature polynomials defined by*

$$\mu(X, \lambda) = \sum_e \mu_e(X)\lambda^e \qquad (e \text{ even}, \ 0 \leq e \leq \dim X) \tag{1.3}$$

satisfy

$$\int \mu(M^p \cap gM^q, \lambda)dg \equiv \mu(M^p, \lambda)\mu(M^q, \lambda) \qquad (\text{mod } \lambda^{p+q-n+1}) \ . \tag{1.4}$$

This version of the kinematic formula shows that the left-hand integral is to some extent independent of n, a fact which was not previously apparent.

Returning to the Weyl expression, we introduce a somewhat different normalization (cf. (2.5))

$$\bar{\mu}_e(X) = \mathcal{O}_{k-e+1}\mu_e(X) \tag{1.5}$$

and corresponding *Weyl curvature polynomials*

(1.6) $$\bar{\mu}(X,\lambda) = \sum_e \bar{\mu}_e(X)\lambda^e \qquad (e \text{ even}, \ 0 \leq e \leq \dim X) .$$

Then we have

Theorem II. *The Weyl curvature polynomials* (1.6) *satisfy*

(1.7) $$\bar{\mu}(X \times Y, \lambda) = \bar{\mu}(X,\lambda)\bar{\mu}(Y,\lambda) ,$$

(1.8) $$V(T_\rho^{(n)}(X)) = \sum_{\substack{0\leq e\leq k \\ e \text{ even}}} B_{n-k+e}(\rho)\bar{\mu}_e(X) ,$$

where $B_m(R)$ *is the volume of the* R*-ball in* E^m, *and* $k = \dim X$.

Note. The numerical coefficients here do not agree with those in (10) of Chern [2]; an error must have slipped in somewhere. See § 4 for details.

2. Some of Chern's formulas

Let X be a k-dimensional Riemann manifold. Then following Chern [2] denote by $\varphi_{\alpha\beta}$ the Levi-Civita connection forms ($1 \leq \alpha, \beta \leq k$), alternating in α and β, and by φ_α ($1 \leq \alpha \leq k$) an orthonormal coframe field. Thus

(2.1) $$d\varphi_\alpha = \sum_\beta \varphi_\beta \wedge \varphi_{\beta\alpha} ,$$

(2.2) $$d\varphi_{\alpha\beta} = \sum_\delta \varphi_{\alpha\delta} \wedge \varphi_{\delta\beta} + \Phi_{\alpha\beta} ,$$

where

(2.3) $$\Phi_{\alpha\beta} = \frac{1}{2} \sum_{\gamma,\delta} S_{\alpha\beta\gamma\delta}\varphi_\gamma \wedge \varphi_\delta .$$

The $S_{\alpha\beta\gamma\delta}$ are components of the curvature tensor, and have the usual properties with respect to the pairs of alternating subscripts (α,β) and (γ,δ). Define for even e, $0 \leq e \leq k$, the pointwise function on X:

(2.4) $$I_e^{(1)} = \frac{(-1)^{e/2}(k-e)!}{2^{e/2}k!} \sum \delta\begin{pmatrix}\alpha_1 & \cdots & \alpha_e \\ \beta_1 & \cdots & \beta_e\end{pmatrix} S_{\alpha_1\alpha_2\beta_1\beta_2} \cdots S_{\alpha_{e-1}\alpha_e\beta_{e-1}\beta_e} ,$$

where $\delta(\ \ )$ is a generalized Kronecker delta equal to ± 1 as the β's are an even or odd permutation of the α's, and zero otherwise; summation is over all α's and β's independently. The numerical factor which precedes $\sum$ in (2.4) was chosen so that $I_e^{(1)} = 1$ when X is the unit sphere S^k in E^{k+1}. (To effect this normalization of Chern's it is necessary to replace the factor $2^{k/2}$ in his (7) by $2^{e/2}$ as in our formula.) Following Chern we define the μ's (we add superscript (1)) as volume integrals:

$$\mu_e^{(1)}(X) = \int_X I_e^{(1)} dv .$$

With this normalization of the μ's, the c's are obtained from Chern's calculation, as follows (he wrote c_i for our $c_{i,j,n,p,q}$). Denote by $\mathcal{O}_m$ the $(m-1)$-dimensional volume of the unit $(m-1)$-sphere in E^m's, so

$$\mathcal{O}_m = 2\pi^{m/2}/\Gamma(m/2)\ , \tag{2.5}$$

and then

$$c_{e-i} = \frac{\mathcal{O}_{n+1}\cdots\mathcal{O}_2\mathcal{O}_{p+q-n+3}\mathcal{O}_{q+2-i}\mathcal{O}_{q+2-e+i}}{\mathcal{O}_{p+2}\mathcal{O}_{p+1}\mathcal{O}_{q+2}\mathcal{O}_{q+1}\mathcal{O}_{p+q-n+3-i}\mathcal{O}_{p+2-n+3-e+i}} b_{e,p+q-n+1-i}\ , \tag{2.6}$$

where the b's are given through an expression denoted by B_e which leads to the formula (Chern's (73) and an integral 13 lines below)

$$\begin{aligned} &\frac{\mathcal{O}_{m-1}\mathcal{O}_m}{2^{m-e}}\int_{-2R}^{2R}(t+1+R^2)^{e/2}(4R^2-t^2)^{(m-e-2)/2}dt \\ &\quad = b_{e,m-e-1}R^{m-e-1}+\cdots+b_{e,m-1}R^{m-1}\ , \\ &\qquad\qquad \text{where } m = p+q-n+2\ . \end{aligned} \tag{2.7}$$

At the end of the paper appears a formula (81)

$$\begin{aligned} &\int \mu_e^{(1)}(M^p \cap E^q)d^{(1)}E^q \\ &\quad = \frac{\mathcal{O}_{n+1}\cdots\mathcal{O}_{n-q+1}}{\mathcal{O}_{q+1}\cdots\mathcal{O}_1}\frac{\mathcal{O}_{p+q-n+2}\mathcal{O}_{p+q-n+1}}{\mathcal{O}_{p+q-n+2-e}}\frac{\mathcal{O}_{p+2-e}}{\mathcal{O}_{p+1}\mathcal{O}_{p+2}}\mu_e^{(1)}(M_p)\ , \end{aligned} \tag{2.8}$$

which refers to the case when gM^q is replaced by the planes E^q of dimension q. Since E^q is not compact, the integration is extended over the space of all q-planes; $d^{(1)}E^q$ is an invariant measure on this space.

3. Calculation of the c's

In his § 7 Chern gave a formula for the b's, hence by implication, for the c's; but the expression is in the form of a *sum*, which is hard to manipulate. Instead, we aim for a *product* expression.

Lemma 1. *Let $e \geq 0$ be even and $r > e-2$. Denote by α_i the coefficient of x^i in*

$$\int_{-1}^{1}(x^2+2ux+1)^{e/2}(1-u^2)^{(r-e)/2}du\ ; \tag{3.1}$$

then

$$c_{i,j,n,p,q} = \frac{\mathcal{O}_{n+1}\cdots\mathcal{O}_2\mathcal{O}_{r+1}\mathcal{O}_{r+2}\mathcal{O}_{r+3}\mathcal{O}_{q+2-e+i}\mathcal{O}_{p+2-i}}{\mathcal{O}_{p+1}\mathcal{O}_{p+2}\mathcal{O}_{q+1}\mathcal{O}_{q+2}\mathcal{O}_{r+3-e+i}\mathcal{O}_{r+3-i}}\alpha_{e-i}\ , \tag{3.2}$$

where $r = p + q - n$.

Proof. Immediate from (2.6), (2.7), and the change of variable $t = 2uR$ in (2.7).

Lemma 2. *Let* α_i *be defined as in Lemma* 1. *Then* $\alpha_i = 0$ *for* i *odd, and for* i *even*

$$\alpha_i = \binom{e/2}{i/2} \frac{\mathcal{O}_{r-i+3}\mathcal{O}_{r-e+i+3}}{\mathcal{O}_{r+3}\mathcal{O}_{r-e+2}} . \tag{3.3}$$

Proof. One can show that

$$\alpha_i = \pi^{\frac{1}{2}} \binom{\frac{e}{2}}{\frac{i}{2}} \frac{\Gamma\left(\frac{r+3}{2}\right)\Gamma\left(\frac{r-e+2}{2}\right)}{\Gamma\left(\frac{r-i+3}{2}\right)\Gamma\left(\frac{r-e+i+3}{2}\right)} ; \tag{3.4}$$

then (3.3) follows by application of (2.5). Formula (3.4) is most easily derived from properties of hypergeometric functions, as was shown by J. van Lint and by J. Boersma. A less elegant method is obtained from

$$\begin{aligned} &\int_{-1}^{1} (x^2 + 2ux + 1)^n (1 - u^2) du \\ &\quad = (x^2 + 1) \int_{-1}^{1} (x^2 + 2ux + 1)^{n-1} (1 - u^2)^s du \\ &\quad - \frac{x}{s+1} \int_{-1}^{1} (x^2 + 2ux + 1)^{n-1} d(1 - u^2)^{s+1} \end{aligned}$$

by completing the started integration by parts and deducing a recurrence relation for $\alpha_i = \alpha_{i,n,s}$. Details on the three methods are found in [3].

Remark. The crucial part of (3.3) is the exact dependence of α_i on r, without which certain vital cancellations could not have taken place. This is reflected in the wording of the problem in [3].

Proof. of Theorem I. The value of $c_{i,j,n,p,q}$ is found from (3.2) and (3.3), and can be written as

$$c_{i,j,n,p,q} = \frac{\mathcal{O}_{n+1} \cdots \mathcal{O}_2 \left(\frac{\mathcal{O}_{r+1}\mathcal{O}_{r+2}(e/2)!}{\mathcal{O}_{r-e+2}} \right)}{\left(\frac{\mathcal{O}_{p+1}\mathcal{O}_{p+2}(i/2)!}{\mathcal{O}_{p-i+2}} \right) \left(\frac{\mathcal{O}_{q+1}\mathcal{O}_{q+2}(j/2)!}{\mathcal{O}_{q-j+2}} \right)} ,$$

where $i + j = e$, $r = p + q - n$. Note that $r = \dim (M^p \cap gM^q)$. Hence the c's become all equal to 1 if we change $d^{(1)}g$ by a factor $(\mathcal{O}_m \cdots \mathcal{O}_2)^{-1}$ and choose $\mu_e(X)$ equal to $\mathcal{O}_{k-e+2}/[\mathcal{O}_{k+1}\mathcal{O}_{k+2}(e/2)!]$ times $\mu_e^{(1)}(X)$; in addition we may introduce a factor $a^{e/2}$, where a is any universal constant. In view of $(e/2)! = 2\pi^{1+e/2}/\mathcal{O}_{e+2}$ we choose $a = \pi$, hence we define

$$\mu_e(X) = \frac{\mathcal{O}_{k-e+2}\mathcal{O}_{e+2}}{2\pi\mathcal{O}_{k+1}\mathcal{O}_{k+2}}\mu_e^{(1)}(X) = \int_X I_e dv\ , \qquad k = \dim X\ , \quad 0 \leq e \leq k\ , \quad e \text{ even,} \tag{3.5}$$

where

$$I_e = \frac{\mathcal{O}_{e+2}}{\mathcal{O}_{k-e+1}}(-1)^{e/2}2^{e/2-1}\pi^{-e/2}\sum \delta\begin{pmatrix}\alpha_i & \cdots & \alpha_e\\ \beta_i & \cdots & \beta_e\end{pmatrix} S_{\alpha_1\alpha_2\beta_1\beta_2}\cdots S_{\alpha_{e-1}\alpha_e\beta_{e-1}\beta_e}\ , \tag{3.6}$$

$$dg = (\mathcal{O}_{n+1}\cdots\mathcal{O}_2)^{-1}d^{(1)}g\ . \tag{3.7}$$

This proves Theorem I. The re-normalization (3.5) also simplifies (2.8); particularly if the measure on the space of q-planes is also re-normalized as

$$dE^q = \frac{\mathcal{O}_{q+1}\cdots\mathcal{O}_1}{\mathcal{O}_{n+1}\cdots\mathcal{O}_{n-q+1}}d^{(1)}E^q\ , \tag{3.8}$$

then the formula becomes

$$\int \mu_e(M^p \cap E^q)dE^q = \mu_e(M^p)\ , \qquad e \leq p + q - n\ , \tag{3.9}$$

or

$$\int \mu(M^p \cap E^q, \lambda)dE^q \equiv \mu(M^p, \lambda) \qquad (\text{mod } \lambda^{p+q-n+1})\ . \tag{3.10}$$

4. The Weyl formula

The volume of an R-ball in E^m is

$$B_m(R) = \int_0^R \mathcal{O}_m r^{m-1}dr = \frac{\mathcal{O}_m}{m}R^m = \frac{\mathcal{O}_{m+2}}{2\pi}R^m\ .$$

The starting point of this section is (1.1), in which we assume the μ's are normalized as in (3.5), (3.6), i.e., by the property

$$\mu_e(S^k(R)) = \frac{\mathcal{O}_{k-e+2}\mathcal{O}_{e+2}}{2\pi\mathcal{O}_{k+2}}R^{k-e} = \frac{\mathcal{O}_{e+2}}{\mathcal{O}_{k+2}}B_{k-e}(R)\ . \tag{4.1}$$

To find the numerical value of the γ's we calculate the volume of the ρ-tube about $S^k(R)$ imbedded in E^n. First $n = k + 1$:

$$B(T_\rho^{(k+1)}(S^k(R))) = \frac{\mathcal{O}_{k+1}}{k+1}((R+\rho)^{k+1} - (R-\rho)^{k+1}) . \tag{4.2}$$

To calculate the volume for $k+1 < n$ we use the following theorem which is an obvious consequence of the possibility to build up ρ-tubes in product situations from products of thin layers of the tubes around the factors.

Theorem III. *Let $X \subset E$ and $Y \subset E^m$ be imbeddings, and $X \times Y \subset E^{n+m}$ the corresponding imbedding of the product. Then*

$$V(T_\rho^{(n+m)}(X \times Y)) = \int_{\substack{\rho_1^2+\rho_2^2 \leq \rho^2 \\ \rho_1, \rho_2 \geq 0}} dV(T_{\rho_1}^{(n)}(X)) \wedge dV(T_{\rho_2}^{(m)}(Y)) . \tag{4.3}$$

In particular, if X and Y are points, we have

$$B_{n+m}(R) = \int_{\substack{\rho_1^2+\rho_2^2 \leq \rho^2 \\ \rho_1, \rho_2 \geq 0}} dB_n(\rho_1) \wedge dB_m(\rho_2) , \tag{4.4}$$

or equivalently,

$$\int_{\substack{\rho_1^2+\rho_2^2 \leq \rho^2 \\ \rho_1, \rho_2 \geq 0}} \rho_1^{n-1}\rho_2^{m-1} d\rho_1 d\rho_2 = \frac{\mathcal{O}_{n+m+2}}{2\pi \mathcal{O}_n \mathcal{O}_m} \rho^{n+m} . \tag{4.5}$$

Note. (4.5) is also easily derived analytically by changing variables: $\rho_1 = r\cos\theta$, $\rho_2 = r \sin\theta$ in the integral and evaluating.

Proof of Theorem II. By applying Theorem III to $X = S^k(R) \subset E^{k+1}$ and Y a single point in E^{n-k-1}, we find

$$\begin{aligned}
V(T_\rho^{(n)}(S^n(R))) &= \int_{\substack{\rho_1^2+\rho_2^2 \leq \rho^2 \\ \rho_1, \rho_2 \geq 0}} d \frac{\mathcal{O}_{k+1}}{k+1} ((R+\rho_1)^{k+1} - (R-\rho_1)^{k+1} \wedge dB_{n-k-1}(\rho_2) \\
&= \int 2\mathcal{O}_{k+1} \sum_{\substack{e \text{ even} \\ 0 \leq e \leq k}} \binom{k}{e} R^{k-e} \rho_1^e d\rho_1 \wedge dB_{n-k-1}(\rho_2) \\
&= 2\mathcal{O}_{k+1} \sum_e \binom{k}{e} R^{k-e} \frac{1}{\mathcal{O}_{e+1}} \int dB_{e+1}(\rho_1) \wedge dB_{n-k-1}(\rho_2) \\
&= \sum_e \binom{k}{e} \frac{2\mathcal{O}_{k+1}}{\mathcal{O}_{e+1}} B_{n-k+e}(\rho) R^{k-e} \\
&= \sum_e \binom{k}{e} \frac{2\mathcal{O}_{k+1}}{\mathcal{O}_{e+1}} \frac{2\pi \mathcal{O}_{k+2}}{\mathcal{O}_{e+2}\mathcal{O}_{k-e+2}} \mu_e(S^k(R)) B_{n-k+e}(\rho) \\
&= \sum_e \mathcal{O}_{k-e+1} \mu_e(S^k(R)) B_{n-k+e}(\rho) .
\end{aligned}$$

In the last step we have used $k!\, \mathcal{O}_{k+1}\mathcal{O}_{k+2} = 2^{-k+2}\pi^{k+1}$, which is just the doubling formula for the Γ-function.

Thus assuming Weyl's basic form of (1.1) is correct we have verified (1.8), and the general formula (1.7) follows easily from Theorem III. In fact, we have

$$V(T_\rho{}^{(n+m)}(X \times Y)) = \sum_e \bar{\mu}_e(X \times Y) B_{n+m-p-q+e}(\rho) ,$$

$$V(T_{\rho_1}^{(n)}(X)) = \sum_i \bar{\mu}_i(X) B_{n-p+i}(\rho_1) ,$$

$$V(T_{\rho_2}^{(m)}(Y)) = \sum_j \bar{\mu}_j(Y) B_{m-q+j}(\rho_2) .$$

Now (4.3) relates the left sides, while (4.4) relates the right sides. It follows that

$$\bar{\mu}_e(X \times Y) = \sum_{\substack{i+j=e \\ i,j \text{ even}}} \bar{\mu}_i(X) \bar{\mu}_j(Y) ,$$

which implies (1.7).

References

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